# Lecture 14: Geometric Buckling and the Reflected Core

CBE 30235: Introduction to Nuclear Engineering — D. T. Leighton

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## Reading Assignment

**Lamarsh & Baratta (4th Edition):**

- **Chapter 6:** Nuclear Reactor Theory.
  - Sections 6.2 – 6.4: Geometric Buckling and Criticality.

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## 1 The Chemical Engineering Analogy: Autocatalysis

Before we solve the reactor equation, let's recognize that this is not a new problem. It is a standard Chemical Engineering problem in disguise.

Consider an **Autocatalytic Reaction** occurring in a gel slab, where a species $A$ catalyzes its own production:

$$A + \ldots \xrightarrow{k} 2A + \ldots$$

The rate of production is proportional to the concentration $C_A$. The steady-state diffusion equation for species $A$ is:

$$\underbrace{\mathcal{D}\nabla^2 C_A}_{\text{Diffusion}} + \underbrace{kC_A}_{\text{Production}} = 0$$

Rearranging:

$$\nabla^2 C_A + \frac{k}{\mathcal{D}} C_A = 0$$

Now look at our **Reactor Equation** from Lecture 13:

$$\nabla^2 \phi + B_m^2 \phi = 0$$

They are identical. A nuclear reactor is simply a diffusion-limited autocatalytic reactor.

- If the slab is too thin, the reactant $A$ (or neutrons) diffuses out the sides faster than it is created. The concentration drops to zero.
- If the slab is thick enough, production exceeds leakage, and the concentration explodes (or sustains itself).

## 2 The Infinite Slab Reactor

Let's solve this for the simplest geometry: an infinite slab of thickness $L$ (from $x = -L/2$ to $x = +L/2$). The reactor equation in 1D is:

$$\frac{d^2\phi}{dx^2} + B_m^2\phi = 0 \tag{1}$$

### 2.1 Boundary Conditions (The Vacuum Approximation)

Neutrons that leave the slab do not come back.

- At the center ($x = 0$), the flux must be symmetric: $\frac{d\phi}{dx} = 0$.
- At the edge ($x = \pm L/2$), the flux must go to zero.
- Note: In the vacuum approximation, we impose zero flux at the boundary; more precisely, the flux extrapolates to zero just outside the physical surface.

### 2.2 The Solution (Sturm-Liouville Problem)

This is a classic eigenvalue problem. The general solution is:

$$\phi(x) = A\cos(B_m x) + C\sin(B_m x)$$

Applying symmetry ($d\phi/dx = 0$ at $x = 0$) kills the sine term ($C = 0$).

$$\phi(x) = A\cos(B_m x)$$

Applying the boundary condition $\phi(L/2) = 0$:

$$\cos\left(B_m \frac{L}{2}\right) = 0$$

For this to be true, the argument must be an odd multiple of $\pi/2$:

$$B_m \frac{L}{2} = \frac{n\pi}{2} \quad \text{for } n = 1, 3, 5\ldots$$

### 2.3 The Fundamental Mode

While mathematical solutions exist for $n = 3, 5, \ldots$, these represent higher harmonic modes.

- **Physical Reality:** You cannot have negative neutron flux. The higher modes ($n > 1$) have negative regions.
- **Time Dependence:** In a real time-dependent solution, while mathematically valid, higher modes also decay faster in time.

Therefore, the only stable, steady-state solution is the **Fundamental Mode** ($n = 1$):

$$B_m = \frac{\pi}{L}$$

Squaring this gives us the **Geometric Buckling ($B_g^2$)** for a slab:

$$B_g^2 = \left(\frac{\pi}{L}\right)^2 \tag{2}$$

# 3 Criticality: The Balance

For the reactor to be critical, the **Material Buckling** (what the fuel *can* do) must equal the **Geometric Buckling** (what the size *requires*):

$$B_m^2 = B_g^2 \tag{3}$$

$$\frac{k_\infty - 1}{L_{diff}^2} = \left(\frac{\pi}{L_{phys}}\right)^2$$

This equation allows you to calculate the Critical Width ($L_{phys}$) required for a given fuel mixture.

# 4 The Reflected Slab (The "Biot Number" Analogy)

Real reactors are surrounded by a **Reflector** (water, graphite, or beryllium) that bounces neutrons back into the core. How do we model this without solving two coupled differential equations? We change the boundary condition.

## 4.1 The Robin Boundary Condition

Instead of the flux dropping to zero at the edge ($\phi = 0$), the reflector imposes a relationship between the flux and its gradient (current).

- By Fick's Law: $J_{out} = -D\frac{d\phi}{dx}$.
- The reflector returns a fraction of this current.

This creates a boundary condition mathematically identical to **Newton's Law of Cooling** in heat transfer:

$$-D\frac{d\phi}{dx}\bigg|_{surface} = h_{eff}(\phi_{surface} - 0) \tag{4}$$

Rearranging to look like a Biot Number condition:

$$\frac{1}{\phi}\frac{d\phi}{dx} = -\frac{h_{eff}}{D} = -\frac{1}{\lambda_{reflector}}$$

## 4.2 The Result: Reflector Savings

If we solve the diffusion equation with this "leaky" boundary condition, the cosine shape doesn't have to hit zero at the wall. It can stay finite.

The effect is that the reactor "feels" bigger than it physically is.

- **Vacuum:** Flux hits zero at $L/2$.
- **Reflected:** Flux would hit zero at some extrapolated distance $L/2 + \delta$.

We call $\delta$ the **Reflector Savings**. It allows us to build the physical core smaller ($L_{new} = L_{old} - 2\delta$) while maintaining the same criticality.

# 5 Geometric Buckling for Other Shapes

We can apply the same method (Helmholtz equation $\nabla^2\phi + B^2\phi = 0$) to other geometries. Rather than deriving the Bessel functions (Cylinder) or Spherical Bessel functions (Sphere) in class, we summarize the results. For a reactor to be critical, $B_m^2$ must equal $B_g^2$.

| **Geometry** | **Dimensions** | **Flux Profile** $\phi(r)$ | **Geometric Buckling** $B_g^2$ |
|---|---|---|---|
| **Infinite Slab** | Thickness $L$ | $\cos(\frac{\pi x}{L})$ | $\left(\frac{\pi}{L}\right)^2$ |
| **Sphere** | Radius $R$ | $\frac{1}{r}\sin(\frac{\pi r}{R})$ | $\left(\frac{\pi}{R}\right)^2$ |
| **Finite Cylinder** | Radius $R$, Height $H$ | $J_0(\frac{2.405r}{R})\cos(\frac{\pi z}{H})$ | $\left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2$ |

Table 1: Geometric Buckling for standard reactor shapes.

# 6 Case Study: The Cecil Kelley Accident (1958)

To understand why Geometric Buckling matters, consider the tragic accident of Cecil Kelley at Los Alamos National Laboratory.

## 6.1 The Setup

Kelley was a chemical operator working with a large mixing tank (1000 liters). The tank contained two immiscible liquids:

1. A heavy aqueous phase (acidic water).
2. A light organic phase (containing dissolved Plutonium) floating on top.

The Plutonium concentration in the organic layer was high ($>$ 3 kg total), but the layer was relatively thin.

- **Geometry:** Slab-like (High Surface Area / Volume ratio).
- **Result:** High leakage. $B_g^2$ (slab) was large. $B_g^2 > B_m^2$. The system was **Sub-critical**.

## 6.2 The Trigger: Turning on the Mixer

When Kelley turned on the stirrer to mix the phases, the fluid dynamics changed the geometry of the plutonium layer.

- The vortex created by the mixer drew the organic layer down into the center of the tank.
- The thin "slab" became a central "blob" (roughly spherical or cylindrical).

## 6.3 The Consequence

- **Geometry Change:** The "blob" had a much lower Surface Area / Volume ratio than the slab.
- **Result:** Neutron leakage dropped dramatically. $B_g^2$ (sphere) is much smaller than $B_g^2$ (slab) for the same volume.
- **Criticality:** Suddenly, $B_g^2 < B_m^2$.

The solution went Prompt Critical. A "Blue Flash" (Cherenkov radiation) was observed. Kelley received a lethal dose of neutrons and gamma rays ($\sim 12,000$ Rads) and died 35 hours later. **Lesson:** Criticality is not just about Mass. It is about **Shape**. A safe slab can become a lethal sphere simply by stirring it.